INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

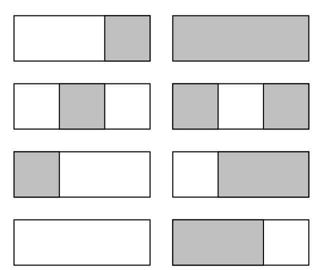
JUNIOR PAPER: YEARS 7, 8, 9, 10

Tournament 43, Northern Fall 2021 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Alice wrote a sequence of $n > 2$ nonzero distinct numbers such that each number in the sequence is greater than the previous one by the same value. Bob wrote the reciprocals of those n numbers in some order. It happens that each number in his sequence is also greater than the previous one by the same value, possibly not the same as in Alice's sequence. What are the possible values of n? (5 points)

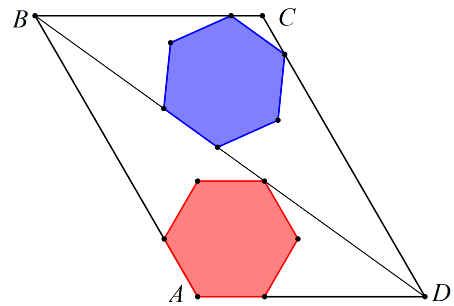
2. On the table there are 8 possible horizontal bars 1×3 such that each 1×1 square is either white or grey (see figure). We are allowed to move them in any direction by any (not necessarily integer) distance. We may not rotate them or turn them over. Is it possible to move the bars so that they do not overlap, all the white points form a polygon bounded by a closed non-self-intersecting broken line and the same is true for all the grey points? (6 points)



3. The hypotenuse of a right-angled triangle has length 1. Consider the line passing through the points of tangency of the incircle with the legs of the triangle. The circumcircle of the triangle cuts out a segment of this line. What are the possible lengths of this segment? (7 points)

4. The number 7 is written on a board. Alice and Bob take turns (Alice begins) writing an additional digit to the number on the board: it is allowed to write the digit at the beginning of the number (provided the digit is nonzero), between any two digits or at the end. If after someone's turn the number on the board is a perfect square then this person wins. Is it possible for one of the players to guarantee the win? (8 points)

5. A diagonal BD splits a parallelogram $ABCD$ with $\angle A = 120°$ into two equal triangles. A regular hexagon is inscribed in the triangle ABD so that two of its consecutive sides lie on AB and AD and one of its vertices lies on BD. Another regular hexagon is inscribed in the triangle CBD so that two of its consecutive vertices lie on CB and CD and one of its sides lies on BD. Which of the hexagons is bigger? (9 points)



6. Let $\lfloor x \rfloor$ denote the integer part of the number x, that is, the largest integer that is not greater than x. Prove that for all positive integers $a_1, a_2, \ldots, a_n$ the following inequality holds true:

$$\left\lfloor \frac{a_1^2}{a_2} \right\rfloor + \left\lfloor \frac{a_2^2}{a_3} \right\rfloor + \ldots + \left\lfloor \frac{a_n^2}{a_1} \right\rfloor \geq a_1 + a_2 + \ldots + a_n.$$

(9 points)

7. 20 buns with jam and 20 buns with treacle are arranged in a row in random order. Alice and Bob take turns taking one bun from either end of the row. Alice starts, and wants to finally obtain exactly 10 buns of both types; Bob tries to prevent this. Is it true that for any order of the buns, Alice can win no matter what Bob does? (12 points)